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On the Equation $\phi(n) = \phi(n + k)$

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Abstract. The number of solutions of the equation $\phi(n) = \phi(n + k)$, for $k \leqq 30$, at intervals of 10^4 to 10^5 are given. The values of n for which $\phi(n) = \phi(n + k) = \phi(n + 2k)$, and for which $\phi(n) = \phi(n + k) = \phi(n + 2k) = \phi(n + 3k)$, are also tabulated.

1. Introduction. In the past few years a number of questions pertaining to the diophantine equation $\phi(n) = \phi(n + k)$ have been asked, where ϕ is the Euler ϕ-function representing the number of integers $\leqq n$ which are prime to n. For example, two such problems are

(1) Does $\phi(n) = \phi(n + 1)$ have infinitely many solutions?

(2) Are there infinitely many solutions of $\phi(n) = \phi(n + 1) = \phi(n + 2)$?

For a more comprehensive list of these problems, we refer the reader to [1].

With the aid of the Number-Divisor Tables of Glaisher, Klee [2] and Moser [3] found for $n \leqq 10^4$ that the equation $\phi(n) = \phi(n + 1)$ has 17 solutions, of which only one solution satisfied the equation $\phi(n) = \phi(n + 1) = \phi(n + 2)$. Recently, we have computed the values of $\phi(n)$ [4], as well as those of the other number-theoretic functions $\sigma(n)$ and $\nu(n)$, for $n \leqq 10^5$. In this note we will examine the results of these calculations as they relate to

$$\phi(n) = \phi(n + k), \qquad k \leqq 30. \tag{1}$$

2. Method. On an IBM 1620, Model 1, the function $\phi(n)$ was computed using the equation

$$\phi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \tag{2}$$

for $n \leqq 10^5$. The solutions of Eq. (1) were counted and the cumulative count of these solutions at intervals of 10^4 to 10^5 are given in Table 1.

3. Summary of Results. The number of solutions varies considerably with k. For example, for $k = 6$, there are 1308 solutions, and for $k = 3$, only 2 solutions, $n \leqq 10^5$. These counts can be categorized into four classes depending upon k.

(a) For $k = 6m$, the number of solutions is relatively very large.

(b) When k is even but not a multiple of 6, the number of solutions is less than for class (a).

(c) When k is odd and not a multiple of 3, the number of solutions is still less.

(d) When $k = 3(2m + 1)$, the number of solutions is lowest and is $\leqq 4$. One solution for any given k = odd is $\phi(k) = \phi(2k)$. The primitive solutions, that is, when n and

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TABLE 1

Number of Solutions of $\phi(n) = \phi(n + k)$; $k \leqq 30$, *at Intervals of* 10^4 *to* 10^5.

n	k = 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
10000	17	84	2	69	4	264	19	64	2	37	22	192	14	55	4
20000	21	141	2	97	5	427	24	84	2	63	23	294	16	85	4
30000	23	182	2	128	9	560	27	99	2	80	27	385	19	109	4
40000	26	229	2	156	10	682	27	114	2	91	30	468	20	130	4
50000	30	261	2	180	11	798	28	134	2	103	33	543	22	152	4
60000	33	297	2	199	11	907	31	148	2	114	35	606	26	173	4
70000	35	334	2	223	11	1022	34	164	2	131	37	667	27	191	4
80000	36	366	2	246	11	1138	35	178	2	143	38	733	27	210	4
90000	36	398	2	261	11	1223	38	189	2	156	39	797	28	237	4
100000	36	434	2	278	11	1308	40	205	2	172	40	851	31	254	4

n	k = 16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
10000	55	23	152	18	56	2	48	26	172	8	35	3	48	26	218
20000	80	27	233	25	88	2	68	29	264	8	56	3	71	33	356
30000	92	30	307	28	115	2	79	31	344	9	67	3	87	36	480
40000	105	32	369	29	136	2	92	35	419	10	77	3	105	39	582
50000	113	35	425	32	157	2	100	38	478	10	85	3	117	41	671
60000	121	40	482	35	177	2	108	41	543	10	93	3	132	43	781
70000	131	43	531	38	195	2	117	47	595	10	99	3	143	47	871
80000	138	43	586	40	210	2	125	49	655	11	107	3	156	51	959
90000	150	48	623	40	230	2	132	50	708	12	117	3	168	54	1038
100000	162	49	665	40	247	2	138	52	763	13	122	3	179	56	1124

k are relatively prime, are for $k = 3$, $\phi(5) = \phi(8)$; for $k = 15$, $\phi(13) = \phi(28)$ and $\phi(17) = \phi(32)$; for $k = 27$, $\phi(55) = \phi(82)$. The solution $(n, k) = (13, 15)$ corresponds to the case for which when $n = 2p - 1$ is a prime and $k = 2p + 1$, where p is an odd prime, then $\phi(n) = \phi(n + k)$. It should be noted that all these solutions have $n < 60$, which is very small compared with the range of investigation. Thus it is very likely that no other solutions exist for these k.

Besides these statistical investigations, it is of interest to identify various values of n which form an arithmetic progression and have the same $\phi(n)$. For arithmetic progressions with a common difference of 1, P. Erdös [5] conjectured that, for every q, $\phi(n) = \phi(n + 1) = \phi(n + 2) = \cdots = \phi(n + q)$ is solvable. In our investigation, we did not find any other arithmetic progression with $q > 2$, and common difference of 1, except the well-known case, i.e., $\phi(5186) = \phi(5187) = \phi(5188)$. However, there are many 3-term arithmetic progressions with a common difference of >1. The lowest values of n, for a given k, for which

$$\phi(n) = \phi(n + k) = \phi(n + 2k) \tag{3}$$

is solvable, are given in Table 2.

TABLE 2

Solutions of $\phi(n) = \phi(n + k) = \phi(n + 2k)$; $k \leqq 30$, $n \leqq 10^5$, *with Least* n.

k	n	$\phi(n)$
1	5186	2592
2	8	4
4	16	8
5	25930	10368
6	72	24
8	32	16
11	57046	25920
12	144	48
14	56	24
16	64	32
17	54778	26880
18	216	72
22	88	40
24	288	96
26	104	48
28	112	48
30	360	96

It should be noted that when k is odd, n is relatively quite large. Furthermore, except for $k = 1$, 5, 11 and 17, arithmetic progressions corresponding to other odd values of k, $k < 30$, are missing. In view of the fact that for $k = 3$, 9, 15, 21 and 27, there are only a few solutions of Eq. (1) as discussed in (d), it is also very likely there are no solutions of Eq. (3) for these k values.

Besides the 3-term arithmetic progressions of n, we also observed some 4-term arithmetic progressions and all of these are recorded in Table 3, where we give n and k for which

$$\phi(n) = \phi(n + k) = \phi(n + 2k) = \phi(n + 3k) \tag{4}$$

is solvable.

It should be noted that in all these cases, k is a multiple of 6. This is perhaps because there are more solutions of Eq. (1) corresponding to these k.

TABLE 3

Solutions of $\phi(n) = \phi(n+k) = \phi(n+2k) = \phi(n+3k)$; $k \leqq 30$, $n \leqq 10^5$.

k	n	$\phi(n)$
6	72	24
6	216	72
6	76326	25440
12	144	48
12	432	144
18	216	72
18	648	216
24	288	96
24	864	288
30	1080	288
30	16200	4320
30	36000	9600
30	48600	12960

4. General Remarks About $\phi(n) = \phi(n+k)$. In general, when n is composite with many distinct prime factors, it is quite difficult to find all solutions of $\phi(n) = \phi(n+k)$. However, when n is a prime or a composite of a special form, it is relatively easy to put restrictions on k such that Eq. (1) holds. For example, the equation $\phi(n) = \phi(n+2)$ is satisfied [3] by $n = 2(2p-1)$ if both p and $2p-1$ are odd primes and by $2^{2^\alpha+1}$ if $2^{2^\alpha}+1$ is a Fermat prime. We give some more such conditions:

(1) If $n = 2^\alpha + 1$ is a prime and $k = 2^{\alpha+1} - n$, then $\phi(n) = \phi(n+k)$. It should be noted that $2^\alpha + 1$ is prime only if $\alpha = 2^\beta$.

(2) If $n = 2^\alpha \cdot 3^\beta + 1$ is a prime and $k = 2^{\alpha+1} \cdot 3^\beta - 1$, then $\phi(n) = \phi(n+k)$.

(3) If $n = 3k$ and 2 and 3 do not divide k, then $\phi(3k) = \phi(4k)$.

(4) If $n = 2p$, where p is an odd prime and $(p+1)/2$ is a prime other than 3, and if $k = 3(p+1)/2 - n$, then $\phi(n) = \phi(n+k)$.

(5) If $n = 3p$, where p is a prime other than 3 and $(p+1)/2$ is a prime other than 5, and if $k = 5(p+1)/2 - n$, then $\phi(n) = \phi(n+k)$.

Similarly, it is possible to write down more of these restrictions on n and k for which Eq. (1) holds.

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